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Near-Optimal Feedback Rendezvous in Elliptic Orbits Accounting for Nonlinear Differential Gravity

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This paper presents a novel approach to the design of near-optimal feedback control laws, for minimum-fuel rendezvous between satellites in elliptic orbits of arbitrary eccentricity. The rendezvous problem for the nonlinear differential gravity model is solved by the application of neighboring optimal feedback control methodology used in conjunction with a nominal trajectory, obtained by solving the related minimum-fuel feedback control problem for the linear Tschauner–Hempel equations, analytically. This novel closed-form solution is used to determine the best values of the final true anomaly by examining its effect on the cost to go for rendezvous. The neighboring feedback control law accounting for nonlinear differential gravity is obtained by using a generalized sweep method, valid when the reference solution does not satisfy the first-order necessary conditions for optimality, exactly. Several numerical examples are analyzed to demonstrate the efficacy of the method.

Introduction

THE problem of satellite rendezvous in orbits around a planet continues to provoke great interest due to its utility in spacecraft servicing, assembly, and inspection. The most basic model for the study of relative motion is given by the Hill–Clohessy–Wiltshire (HCW) equations [1,2]. These equations model relative motion between a chaser and target vehicle under the assumptions of circular target orbit, linear differential gravity field, and two-body dynamics. The HCW equations constitute a sixth-order, linear model, which is extremely useful for preliminary analysis. However, the scope of this model is severely limited and it is unreliable when the target orbit is eccentric. The distance between the chaser and target is not negligible when compared to that of the target from the gravitational center. To study the optimal rendezvous problem rigorously, it is necessary to model eccentricity and nonlinearity effects. A vast body of literature exists on the study of relative motion dynamics for the two-body problem when the HCW assumptions are violated.

Knollman and Pyron [3] and London [4] obtained approximate solutions to the HCW equations, perturbed by second-order nonlinearities. Karlgaard and Lutze [5] also obtained analytical relative motion equations near a circular orbit that are correct to second order, using spherical coordinates. Richardson and Mitchell [6] used a perturbation analysis to obtain the solution valid for third-order nonlinearities by enforcing periodicity conditions on the linear solution.

The effect of eccentricity on the relative motion equations has been studied both for the linear as well as nonlinear equations. The linear problem for eccentric reference orbits was introduced by Tschauner and Hempel [7]. The Tschauner–Hempel (TH) equations use true anomaly of the target as the independent variable, rather than time, and the local position of the chaser is normalized by the radial distance of the target. Analytical expressions for relative motion were obtained by de Vries [8], who treated eccentricity as a perturbation to the TH equations. In his approach, only terms of up to first order in eccentricity were considered. The TH equations by themselves admit analytical solutions in the form of special integrals as shown in [9–11]. These solutions are valid for arbitrary eccentricities and have been used for the determination of state transition matrices for linearized relative motion with true anomaly as the independent variable [12–15]. Melton [16] and Broucke [17] have developed state transition matrices for relative motion with time as the independent variable. Melton [16] uses a series expansion for radial distance and true anomaly in terms of time. However, for moderate eccentricities, the convergence of such series requires the inclusion of higher-order terms. Attempts were made by Euler and Shulman [18] to obtain solutions to the TH equations with second-order gravitational perturbations to enhance their domain of applicability to large relative distances and arbitrary eccentricity. Although the authors noted the nonexistence of an analytical solution to this problem at that time, in a recent work [19], such a solution has been presented for the special case of periodic motion.

The optimal rendezvous problem is also of historical interest, especially due to Lawden's primer vector theory [9]. Billik [20] used a differential games approach to design optimal thruster programming laws for the HCW equations. Euler [21] approached the rendezvous problem by attempting to find an open-loop optimal control to the TH equations, for the standard quadratic cost function valid for power-limited, low-thrust propulsion. However, a complete analytical solution could not be found, and results were obtained by restricting the equations to first order in eccentricity. Later work by Carter [22] reduced Euler's problem to the numerical evaluation of one key matrix. Edelbaum [23] formulated and solved the optimal rendezvous problem in terms of small orbital element differences. Gobetz [24] also used a similar linearization in orbital element space, with the additional assumption of a near-circular target orbit, but used a nonsingular element set that extended the validity of the laws to those cases where eccentricity and inclination are zero—known singularities in the classical orbital element set. The elements used by Gobetz are similar to the equinoctial elements [25]. Jezewski and Stoolz [26] formulated the constant-thrust orbital transfer problem, by expressing the gravity field as a third-order polynomial in time by using two measurements of position and velocity and solving for the polynomial coefficients. Solutions to the continuous-thrust optimal rendezvous problem in a linearized gravity field using the TH equations have been explored extensively in [10,27,28]. In recent work by Zanon and Campbell [29,30], an approximate solution for an open-loop, bounded-input controller was developed using spline function approximations for certain key integrals. In [30], a similar approach was discussed with spacecraft attitude constraints. Palmer

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[31] presented an analytical formulation for optimal transfer paths based on the HCW equations. Breger and How [32] examined several online convex optimization formulations based on linear programming techniques for docking and autonomous fuel-optimized spacecraft rendezvous trajectories.

Various methods for the treatment of nonlinearity in the optimal rendezvous problem have been proposed. Williams [33] presented a quasilinearization scheme for spacecraft rendezvous on small relative inclination orbits by using tethers. Park et al. [34] proposed a feedback controller for the nonlinear optimal rendezvous of a spacecraft near a circular orbit by using the Hamilton–Jacobi theory. Kim and Spencer [35] demonstrated the use of genetic algorithms to solve for Hohmann and bielliptical transfers.

The study of the literature on the optimal rendezvous problem reveals that analytical solutions for arbitrary eccentricity as well as nonlinear differential gravity have not been obtained. The necessity for such solutions arises from recent interest in formations in highly elliptic orbits, such as the magnetosphere multiscale mission [36,37]. Moreover, the solutions to the optimal rendezvous problem obtained by accounting for the above-mentioned perturbations will typically require lower costs because the physics of the problem is accurately reflected in the model.

This paper begins by presenting the nonlinear TH equations in nondimensional form where a perturbation parameter, which captures the effect of nonlinearity as well as eccentricity of the reference orbit, is identified. Next, a novel analytical solution is presented for the minimum-fuel rendezvous problem for the nonautonomous linear model obtained by setting the perturbation parameter to zero. Finally, the neighboring optimal control problem is posed for nonzero values of the perturbation parameter. Its solution, accurate to first order, is obtained in feedback form [38–40] to meet the terminal constraint for rendezvous accurately. The results are validated on the fully nonlinear model. Several examples for various combinations of eccentricity and initial separation distance are presented to demonstrate the excellent performance and wide applicability of the proposed methodology.

Equations of Motion

The nonlinear (dimensional) rendezvous equations are

$$\ddot\xi + 2\dot\theta\,\dot\chi - \dot\theta^2\xi + \ddot\theta\chi = -\frac{\mu(r+\xi)}{[(r+\xi)^2+\chi^2+\zeta^2]^{3/2}} + \frac{\mu}{r^2} + U_\xi \quad (1)$$

$$\ddot\chi + 2\dot\theta\,\dot\xi - \dot\theta^2\chi + \ddot\theta\xi = -\frac{\mu\chi}{[(r+\xi)^2+\chi^2+\zeta^2]^{3/2}} + U_\chi \quad (2)$$

$$\ddot\zeta = -\frac{\mu\zeta}{[(r+\xi)^2+\chi^2+\zeta^2]^{3/2}} + U_\zeta \quad (3)$$

where ξ, χ, and ζ indicate, respectively, the radial, along-track, and out-of-plane components of the position vector of the chaser satellite in the target satellite's local–vertical–local–horizontal (LVLH) frame; r is the radial distance of the target from the center of the attracting body, θ is defined as the true argument of latitude (sum of the argument of the periapsis and true anomaly), and U_ξ, U_χ, and U_ζ indicate the components of the control acceleration vector along the respective directions. In the above equations, μ is a gravitational parameter and time is the independent variable. Furthermore, $(\dot{\ })$ and $(\ddot{\ })$, respectively, denote the first and second derivatives with respect to time. To convert the rendezvous equations into the TH form, the following steps are performed:

1) The independent variable is changed from time to true anomaly, denoted by f. Therefore,

$$(\dot{\ }) = \dot f(0000') = \bar n(1+e\cos f)^2(0000') \quad (4)$$

$$\begin{aligned}(\ddot{\ }) &= \dot f^2(0000'') + \ddot f(0000') \\ &= \bar n^2(1+e\cos f)^3[(1+e\cos f)(0000'') - 2e\sin f(0000')]\end{aligned} \quad (5)$$

where $\bar n = \sqrt{\mu/p^3}$, $p = a(1-e^2)$ is the semiparameter of the target orbit, a is the semimajor axis, e is the eccentricity of the target orbit, and $(')$ and $('')$ denote the first and second derivatives with respect to f, respectively.

2) Next, the nondimensional position variables are defined as follows: $x = (1+e\cos f)\xi/\rho_0$, $y = (1+e\cos f)\chi/\rho_0$, and $z = (1+e\cos f)\zeta/\rho_0$, where ρ_0, the size of the relative orbit, is used as a characteristic length. The nondimensional control acceleration components are given by the following: $u_x = U_\xi/(\rho_0\bar n^2)$, $u_y = U_\chi/(\rho_0\bar n^2)$, and $u_z = U_\zeta/(\rho_0\bar n^2)$.

3) Finally, let $\varepsilon = \rho_0/p$ be the nonlinearity measure of the system. The degree of nonlinearity not only depends on the size of the relative orbit, but also on the eccentricity of the reference orbit, introduced through the semiparameter.

After performing the above steps, the state-space representation of the TH equations can be obtained with $\mathbf{x} \in \mathbb{R}^{6\times 1} \equiv [x; y; z; x'; y'; z']^T$, as follows:

$$\mathbf{x}' = \mathbf{h}(\mathbf{x}, f) + B(f)\mathbf{u} \quad (6)$$

where $\mathbf{u} \in \mathbb{R}^{3\times 1} \equiv [u_x; u_y; u_z]^T$. In more explicit form, the components of the above vector equation can be written as

$$x' = v_1 \quad (7)$$

$$y' = v_2 \quad (8)$$

$$z' = v_3 \quad (9)$$

$$\begin{aligned}v_1' &= 2v_2 + \frac{x}{(1+e\cos f)} + \frac{1}{(1+e\cos f)}\frac{1}{\varepsilon} \\ &\quad \times \left\{1 - \frac{(1+\varepsilon x)}{[(1+\varepsilon x)^2 + \varepsilon^2 y^2 + \varepsilon^2 z^2]^{3/2}}\right\} + \frac{u_x}{(1+e\cos f)^3}\end{aligned} \quad (10)$$

$$\begin{aligned}v_2' &= 2v_1 + \frac{y}{(1+e\cos f)} \\ &\quad + \frac{1}{(1+e\cos f)}\frac{y}{[(1+\varepsilon x)^2 + \varepsilon^2 y^2 + \varepsilon^2 z^2]^{3/2}} + \frac{u_y}{(1+e\cos f)^3}\end{aligned} \quad (11)$$

$$\begin{aligned}v_3' &= \frac{e\cos f}{(1+e\cos f)}z \\ &\quad - \frac{1}{(1+e\cos f)}\frac{z}{[(1+\varepsilon x)^2 + \varepsilon^2 y^2 + \varepsilon^2 z^2]^{3/2}} + \frac{u_z}{(1+e\cos f)^3}\end{aligned} \quad (12)$$

Performance Index

Before proceeding further, it is noted that the integral sum of squares of the control accelerations is used in this paper as the performance index for the minimum-fuel formulation. The chosen form of the performance index is appropriate for power-limited, low-thrust propulsion [41,42]. For variable specific impulse (I_{sp}) power-limited propulsion systems, the fuel consumed is directly proportional to the quadratic performance index used in this paper [42] but with time as the independent variable. However, in this paper, true anomaly is used as the independent variable instead of time, because it allows for a closed-form solution to the related nominal option control problem. As is shown in the following sections, the change in the independent variable also results in the benefit of storage of gain data at a nonuniform rate in time and naturally penalizes the control effort near perigee.

Nominal Solution

A key component of the work that is reported in this paper is the determination of the analytical reference solution to a special case of the main problem. This reference solution and a special choice of a perturbation parameter lead to an accurate solution to the main optimal control problem. A good reference solution is essential for fast convergence of Taylor series expansions about it.

The next section provides the closed-form solution for the nominal linear-quadratic (LQ) optimal control problem.

Analytical Solution for the LQ Problem

The linear part of the TH equations can be obtained by taking the limit of Eq. (6) as $\varepsilon \to 0$. The linear TH equations in state-space representation are given as

$$\mathbf{x}' = A(f)\mathbf{x} + B(f)\mathbf{u} \tag{13}$$

where $\mathbf{x} \in \mathbb{R}^6$, $u \in \mathbb{R}^3$, A: $\mathbb{R}_{\geq 0} \to \mathbb{R}^{6\times 6}$, B: $\mathbb{R}_{\geq 0} \to \mathbb{R}^{6\times 3}$, and

$$\mathbf{x} = [x; y; z; x'; y'; z']^T, \qquad B(f) = (1 + e\cos f)^{-3}\begin{bmatrix} 0_{3\times 3} \\ 1_{3\times 3} \end{bmatrix}$$

$$A(f) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{3}{(1+e\cos f)} & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

Note that the actual control acceleration $\mathbf{U}$ is obtained from

$$\mathbf{U}(f) = \rho_0 \bar{n}^2 \mathbf{u}(f) \tag{15}$$

The LQ optimal control problem is posed as follows:

Minimize:

$$J = \frac{1}{2}\int_{f_0}^{f_T} (\mathbf{u}^T R \mathbf{u})\,\mathrm{d}f \tag{16}$$

subject to Eq. (13) and initial and final conditions as

$$\mathbf{x}(f_0) = \mathbf{x}_0; \qquad \mathbf{x}(f_T) = \mathbf{x}_T \tag{17}$$

Note that the weight matrix R in Eq. (16) is positive definite. For the LQ problem posed, it can be shown that the necessary conditions for optimality [43] yield the following relations for the controls and costates, denoted by $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda}' = -A^T\boldsymbol{\lambda}; \qquad \mathbf{u} = -R^{-1}B^T\boldsymbol{\lambda} \tag{18}$$

The solution to the augmented linear system composed of states and costates can be obtained using the state transition matrix (STM), Φ as follows:

$$\begin{bmatrix} \mathbf{x}' \\ \boldsymbol{\lambda}' \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ 0_{6\times 6} & -A^T \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{x}(f) \\ \boldsymbol{\lambda}(f) \end{bmatrix} = \Phi(f, f_0)\begin{bmatrix} \mathbf{x}(f_0) \\ \boldsymbol{\lambda}(f_0) \end{bmatrix} \tag{19}$$

Consequently, the initial values of $\boldsymbol{\lambda}$ can be determined as follows:

$$\Phi = \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \Rightarrow \boldsymbol{\lambda}(f_0) \equiv \boldsymbol{\lambda}_0 = \Phi_{x\lambda}^{-1}(f_T, f_0)[\mathbf{x}_T - \Phi_{xx}(f_T, f_0)\mathbf{x}_0] \tag{20}$$

Thus, the LQ problem can be solved if Φ_{xx}, $\Phi_{\lambda\lambda}$, and $\Phi_{x\lambda}$ are determined.

Although not immediately obvious, the required submatrices can be evaluated analytically by using the closed-form solutions of the TH equations. The solutions to the unforced TH equations are given in [14],

$$x(f) = c_1\cos f(1 + e\cos f) + c_2\sin f(1 + e\cos f) + \frac{2c_3}{\eta^2}\left[1 - \frac{3e}{2\eta^3}\sin f(1 + e\cos f)K(f)\right] \tag{21}$$

$$y(f) = -c_1\sin f(2 + e\cos f) + c_2\cos f(2 + e\cos f) - \frac{3c_3}{\eta^5}(1 + e\cos f)^2 K(f) + c_4 \tag{22}$$

$$z(f) = c_5\cos f + c_6\sin f \tag{23}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are constants of integration, $\eta = \sqrt{1 - e^2}$, and

$$K(f) \equiv \int_{f_0}^{f} \frac{\eta^3}{(1 + e\cos f)^2}\,\mathrm{d}f = (E - e\sin E) - (E_0 - e\sin E_0) = n\Delta t \tag{24}$$

The function $K(f)$, denoting scaled time since the epoch, is easier to evaluate in terms of the eccentric anomaly E. The following equations show the relationships between eccentric and true anomalies [44]:

$$\cos f = \frac{\cos E - e}{1 - e\cos E}, \qquad \sin f = \frac{\eta\sin E}{1 - e\cos E}$$
$$\cos E = \frac{\cos f + e}{1 + e\cos f}, \qquad \sin E = \frac{\eta\sin f}{1 + e\cos f} \tag{25}$$

The following expressions are also provided in [14]:

$$x'(f) = -c_1(\sin f + e\sin 2f) + c_2(\cos f + e\cos 2f) - \frac{3ec_3}{\eta^2}\left[\frac{\sin f}{1 + e\cos f} + \frac{1}{\eta^3}(\cos f + e\cos 2f)K(f)\right] \tag{26}$$

$$y'(f) = -c_1(2\cos f + e\cos 2f) - c_2(2\sin f + e\sin 2f) - \frac{3c_3}{\eta^5}[\eta^3 - e(2\sin f + e\sin 2f)K(f)] \tag{27}$$

$$z'(f) = -c_5\sin f + c_6\cos f \tag{28}$$

Calculations are simplified if the in-plane problem, pertaining to motion in the x and y directions, and the out-of-plane problem, pertaining to motion in the z direction, are treated separately. As shown in [19], the relationship between the states and the integration constants can be expressed as $\mathbf{x} = L(f)\mathbf{c}$, where the entries of the matrix L can be obtained from the terms corresponding to the constants of integration in Eqs. (21–23) and (26–28), with the notation $\mathbf{c} = [c_1, c_2, c_3, c_4, c_5, c_6]^T$. This matrix has determinant equal to 1, and the constants of integration can then be calculated by its adjoint, in terms of the initial condition vector $\mathbf{x}_0$, as

$$c_1 = -\frac{3}{\eta^2}(e + \cos f_0)x_0 - \frac{1}{\eta^2}\sin f_0(1 + e\cos f_0)x'_0 - \frac{1}{\eta^2}(e + 2\cos f_0 + e\cos^2 f_0)y'_0 \tag{29}$$

$$c_2 = -\frac{3}{\eta^3}\frac{\sin f_0(1 + e\cos f_0 + e^2)}{(1 + e\cos f_0)}x_0 + \frac{1}{\eta^2}(\cos f_0 - 2e + e\cos f_0)x_0' - \frac{1}{\eta^2}\sin f_0(2 + e\cos f_0)y_0' \tag{30}$$

$$c_3 = l_1 x_0 + l_2 x_0' + l_3 y_0' \tag{31}$$

$$c_4 = -\frac{1}{\eta^2}(2 + e\cos f_0)\left[\frac{3e\sin f_0}{(1 + e\cos f_0)}x_0 + (1 - e\cos f_0)x_0' + e\sin f_0 y_0'\right] + y_0 \tag{32}$$

$$c_5 = (\cos f_0)z_0 - (\sin f_0)z_0' \tag{33}$$

$$c_6 = (\sin f_0)z_0 + (\cos f_0)z_0' \tag{34}$$

where

$$l_1 = 2 + 3e\cos f_0 + e^2 \qquad l_2 = e\sin f_0(1 + e\cos f_0) \qquad l_3 = (1 + e\cos f_0)^2 \tag{35}$$

Equations (29–34) may be rewritten as $\mathbf{c} = M(f_0)\mathbf{x}_0$, where $M = \mathrm{adj}(L)$. Using the above equations, the solution for Φ_{xx} can be written as

$$\mathbf{x}(f) = L(f)M(f_0)\mathbf{x}_0 \Rightarrow \Phi_{xx}(f, f_0) = L(f)M(f_0) \tag{36}$$

The remaining blocks of the STM can be obtained by observing that the Hamiltonian system, Eq. (19), leads to a state transition matrix that is symplectic in nature. More specifically, if $\mathfrak{J}$ denotes the symplectic matrix of appropriate order, Eqs. (19) and (20) imply

$$\Phi \mathfrak{J} \Phi^T = \mathfrak{J}$$

or

$$\begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix}\begin{bmatrix} 0_{6\times6} & I_{6\times6} \\ -I_{6\times6} & 0_{6\times6} \end{bmatrix}\begin{bmatrix} \Phi_{xx}^T & \Phi_{\lambda x}^T \\ \Phi_{x\lambda}^T & \Phi_{\lambda\lambda}^T \end{bmatrix} = \begin{bmatrix} 0_{6\times6} & I_{6\times6} \\ -I_{6\times6} & 0_{6\times6} \end{bmatrix} \tag{37}$$

Resolving the matrix multiplication and comparing the block matrices on both sides leads to four equations, one of which is given as

$$\Phi_{xx}\Phi_{\lambda\lambda}^T - \Phi_{x\lambda}\Phi_{\lambda x}^T = I_{6\times6} \tag{38}$$

If $\mathbf{x}$ is not present in the cost function, Eq. (16) (as is the case considered here), then $\boldsymbol{\lambda}'$ does not depend on $\mathbf{x}$ explicitly. In such a case, $\Phi_{\lambda x} = 0_{6\times6}$ and it follows that

$$\Phi_{\lambda\lambda}(f, f_0) = \Phi_{xx}^T(f_0, f) = M^T(f)L^T(f_0) \tag{39}$$

From Eqs. (10–12) and (19), the solution to the forced system is

$$\begin{aligned}\mathbf{x}(f) &= \Phi_{xx}(f, f_0)\mathbf{x}_0 - \int_{f_0}^{f}\Phi_{xx}(f, s)B(s)R^{-1}B^T(s)\Phi_{\lambda\lambda}(s, f_0)\boldsymbol{\lambda}_0\,\mathrm{d}s \\ &= \Phi_{xx}(f, f_0)\mathbf{x}_0 + \Phi_{x\lambda}(f, f_0)\boldsymbol{\lambda}_0\end{aligned} \tag{40}$$

Hence,

$$\Phi_{x\lambda}(f, f_0) = -\int_{f_0}^{f}\Phi_{xx}(f, s)B(s)R^{-1}B^T(s)\Phi_{\lambda\lambda}(s, f_0)\boldsymbol{\lambda}_0\,\mathrm{d}s \tag{41}$$

It can be shown by substituting Eq. (39) into Eq. (41) that

$$\begin{aligned}\Phi_{x\lambda}(f, f_0) &= -L(f)\left(\int_{f_0}^{f}M(s)B(s)R^{-1}B^T(s)M^T(s)\,\mathrm{d}s\right)L^T(f_0) \\ &= -L(f)[N(f) - N(f_0)]L^T(f_0)\end{aligned} \tag{42}$$

where

$$N(f) = \int M(f)B(f)R^{-1}B^T(f)M^T(f)\,\mathrm{d}f \tag{43}$$

It is evident that the problem can be solved completely if $N(f)$ is evaluated. Note that $N(f)$ is symmetric and it can be shown that

$$B(f)R^{-1}B^T(f) = (1 + e\cos f)^{-6}\begin{bmatrix} 0 & 0 \\ 0 & R^{-1} \end{bmatrix} \tag{44}$$

Though it appears at the outset that the integration process is complicated due to the presence of terms containing $(1 + e\cos f)$ in the denominator, integration can be performed by changing the independent variable to E; the results are given in Appendix A. Consequently, $N(f)$ may be rewritten as

$$N(f) = N^{(3)}(f)K^3(f) + N^{(2)}(f)K^2(f) + N^{(1)}(f)K(f) + N^{(0)}(f) \tag{45}$$

The components of $N^{(0,\ldots,3)}(f)$ are presented in Appendix A. For the sake of brevity the following definition is used:

$$\bar{N}(f_T, f_0) \triangleq N(f_T) - N(f_0) \tag{46}$$

Dependence of the Cost to Go on the Final Value of True Anomaly

For given initial and final conditions, $\mathbf{x}_0$ and $\mathbf{x}_T$, and epoch f_0, the cost is a function of the final value of true anomaly, f_T. Using Eqs. (16) and (18), we obtain the following representation of the cost function:

$$\begin{aligned}J^*(f_T) &= \frac{1}{2}\int_{f_0}^{f_T}\boldsymbol{\lambda}^T B(f)R^{-1}B(f)^T\boldsymbol{\lambda}\,\mathrm{d}f \\ &\Rightarrow \frac{1}{2}\int_{f_0}^{f_T}\boldsymbol{\lambda}_0^{\mathbf{T}}\Phi_{\lambda\lambda}^T(f, f_0)B(f)R^{-1}B(f)^T\Phi_{\lambda\lambda}^T(f, f_0)\boldsymbol{\lambda}_0\,\mathrm{d}f\end{aligned} \tag{47}$$

The following definitions are used for ease of notation: $L(f_0) = L_0$, $L(f_T) = L_T$, $M(f_0) = M_0$, and $M(f_T) = M_T$. Furthermore, because $\boldsymbol{\lambda}_0$ is a constant, the use of Eqs. (39), (43), and (46) results in the following:

$$J^* = \tfrac{1}{2}\boldsymbol{\lambda}_0^T L_0 \bar{N} L_0^T \boldsymbol{\lambda}_0 \tag{48}$$

Using Eqs. (20) and (42) yields the following expression for the cost:

$$J^* = \tfrac{1}{2}(M_T\mathbf{x}_T - M_0\mathbf{x}_0)^T\bar{N}^{-1}(M_T\mathbf{x}_T - M_0\mathbf{x}_0) \tag{49}$$

Because $J^* \geq 0$ for all possible choices of initial and final conditions, it follows that $\bar{N}^{-1} \geq 0$. In fact, $J^* = 0$ implies that either 1) $M_T\mathbf{x}_T = M_0\mathbf{x}_0$, or 2) $\bar{N}^{-1}$ has at least one eigenvalue that is zero. The first case is possible only if $\mathbf{x}_T = \Phi_{xx}(f_T, f_0)\mathbf{x}_0$, or if the desired final states arise from the natural evolution of the system from the initial states. The second condition implies that at least one eigenvalue of $\bar{N}$ is infinity. Because $\bar{N}$ comprises only bounded periodic terms and an increasing function $K(f)$, this must mean $K(f_T) \to \infty$ and consequently, $f_T \to \infty$. This is also intuitive from the physics of the problem, because in both cases, the control requirement is zero. Therefore, in all other cases where nonzero control is required, $\{\bar{N}, \bar{N}^{-1}\} > 0$. Furthermore, because $\bar{N}$ is symmetric and block diagonal in structure, considerable computation time can be saved by the use of methods such as the Choleskey decomposition to find its inverse.

Feedback Rendezvous Scheme for the Nonlinear System

This section of the paper deals with the main problem of interest. The rendezvous problem is solved as a neighboring optimal control problem (OCP) using the closed-form reference solution developed in the previous section. Issues such as sensor and actuator modeling, noise and filtering, plume impingement, etc., are beyond the scope of this paper. In essence, the closed-form reference solution for $\varepsilon = 0$ is used to compute the feedback control solution for the case when $\varepsilon \neq 0$ by formulating a nonautonomous LQ problem with terminal constraints. The LQ problem can be solved either by using a sweep method and storing the required gain matrices or by repeatedly solving the resulting Hamiltonian system (in real time) for the costates at the current time. The solution for the current costate vector is used to compute the current control command. The second approach is more direct if the computation indeed can be achieved with speed and accuracy, as the need for storage of the time-varying gain matrices is eliminated. An example of the second method of solution using the pseudospectral method is discussed in [45]. In this paper, the first approach, that is, the sweep method is treated in a novel framework.

Continuous Control

As mentioned before, the focus of this paper is the solution of the OCP involving the TH equations, Eq. (6) with $\varepsilon \neq 0$. The Hamiltonian can be written as

$$H = \tfrac{1}{2}\mathbf{u}^T R\mathbf{u} + \boldsymbol{\lambda}^T[\mathbf{h}(\mathbf{x}, f) + B(f)\mathbf{u}] \tag{50}$$

The necessary conditions for optimality are

$$\boldsymbol{\lambda}' = -H_x \tag{51}$$

$$\mathbf{x}' = H_\lambda \tag{52}$$

and

$$H_u = 0 \tag{53}$$

Let $\mathbf{x}_{\text{ref}}$, $\mathbf{u}_{\text{ref}}$, and $\boldsymbol{\lambda}_{\text{ref}}$ indicate the reference state, control, and costate vectors, respectively.

To facilitate the near-optimal feedback solution to the first order, Eqs. (51–53) can be approximated by the following linear differential equations where all the partial derivatives are taken along the reference trajectory:

$$\mathbf{x}' = H_{\lambda x}\mathbf{x} + H_{\lambda u}\mathbf{u} + \mathbf{m}_1 \tag{54}$$

$$\boldsymbol{\lambda}' = -[H_{xx}\mathbf{x} + H_{xu}\mathbf{u} + H_{x\lambda}\boldsymbol{\lambda} + \mathbf{m}_2] \tag{55}$$

$$\mathbf{u} = -H_{uu}^{-1}[H_{ux}\mathbf{x} + H_{u\lambda}\boldsymbol{\lambda} + \mathbf{m}] \tag{56}$$

where $\mathbf{m}_1$, $\mathbf{m}_2$, and $\mathbf{m}$ are functions of true anomaly, given as

$$\mathbf{m}_1 = H_\lambda - H_{\lambda x}\mathbf{x}_{\text{ref}} - H_{\lambda u}\mathbf{u}_{\text{ref}} \tag{57}$$

$$\mathbf{m}_2 = H_x - H_{xx}\mathbf{x}_{\text{ref}} - H_{xu}\mathbf{u}_{\text{ref}} - H_{x\lambda}\boldsymbol{\lambda}_{\text{ref}} \tag{58}$$

$$\mathbf{m} = H_u - H_{ux}\mathbf{x}_{\text{ref}} - H_{uu}\mathbf{u}_{\text{ref}} - H_{u\lambda}\boldsymbol{\lambda}_{\text{ref}} \tag{59}$$

Analytical expressions for the partial derivatives are given in Appendix B. It can be noted that the higher-order terms in the Taylor series expansion of Eqs. (51–53) about the nominal are of the order of the perturbation parameter ε and can be neglected as the order increases. In fact, the availability of the excellent reference solution makes ε small for many realistic cases of practical importance. Hence, the first-order correction to the feedback control suffices well to obtain the desired accuracy at the terminal.

Near-Optimal Feedback Solution

The feedback solution for the linear two-point boundary value problem defined by Eqs. (54–59) can be constructed by using the backward sweep method in a way similar to that given in [42]. The key steps are the substitutions

$$\boldsymbol{\lambda} = S\mathbf{x} + P\boldsymbol{\nu} + G \tag{60}$$

where ν is a terminal Lagrange multiplier associated with the terminal constraint

$$\psi = C\mathbf{x}(f) \tag{61}$$

and the following representation of ψ:

$$\psi = P^T\mathbf{x} + V\boldsymbol{\nu} + \underline{L} \tag{62}$$

where S, P, V, $\underline{L}$, and G are true anomaly-dependent gain matrices. Equation (62) is used to eliminate $\boldsymbol{\nu}$ in terms of the states. Using Eqs. (60–62) in Eqs. (54–56) and following the application of the sweep method, the near-optimal feedback control law can be written as

$$\begin{aligned}\mathbf{u}^* = {}& -H_{uu}^{-1}[[H_{u\lambda}\{S - PV^{-1}P^T\} + H_{ux}]\mathbf{x} \\ & + H_{u\lambda}[PV^{-1}(\psi - \underline{L}) + G] + \mathbf{m}]\end{aligned} \tag{63}$$

The differential equations for the gains are given as

$$\begin{aligned}& S' + S\underline{A} + \underline{A}^T S + S\underline{C}S + \underline{D} = 0 \quad \text{(Riccati equation)} \\ & P' + [S\underline{C} + \underline{A}^T]P = 0 \qquad G' + [S\underline{C} + \underline{A}^T]G + \underline{K} = 0 \\ & V' + P^T\underline{C}P = 0 \qquad \underline{L}' + P^T[\underline{C}G + \mathbf{m}_1 - H_{\lambda u}H_{uu}^{-1}\mathbf{m}] = 0\end{aligned} \tag{64}$$

where

$$\begin{aligned}& \underline{A} = H_{\lambda x} - H_{\lambda u}H_{uu}^{-1}H_{ux} \qquad \underline{C} = -H_{\lambda u}H_{uu}^{-1}H_{u\lambda} \\ & \underline{D} = H_{xx} - H_{xu}H_{uu}^{-1}H_{ux} \\ & \underline{K} = S\mathbf{m}_1 + \mathbf{m}_2 - [H_{xu} + SH_{\lambda u}]H_{uu}^{-1}\mathbf{m}\end{aligned} \tag{65}$$

The boundary conditions for these equations are given by the transversality conditions [43] as follows:

$$\begin{aligned}& S(f_T) = 0; \qquad P(f_T) = C^T; \qquad G(f_T) = 0 \\ & V(f_T) = 0; \qquad \underline{L}(f_T) = 0\end{aligned} \tag{66}$$

The differential equations for computing the gains as shown in Eqs. (64) can be integrated backward, with f as the independent variable. In this work, the gain equations were integrated within an accuracy of 10^{-9} by using the variable step-size fourth-order Runge–Kutta method, as implemented in the MATLAB®[§] function "ode45." The results were stored and used for the propagation of the closed-loop nonlinear system. Cubic spline interpolation is used to provide the values of the gain matrices stored at a finite number of true anomaly values. As mentioned before, for eccentric orbits, the use of uniform samples in true anomaly leads to nonuniform samples in time. The terminal error in rendezvous is a function of the number of samples used for storing the gains.

Numerical Results and Discussion

The availability of an analytical solution to the LQ problem for $\varepsilon = 0$ leads to the examination of the dependence of the cost to go on the final true anomaly. Because the control and costates are known analytically, the Hamiltonian of the system reduces to a function of one implicit variable f. Therefore, solving for the zeros of the

[§]MATLAB 7.0, data available online at http://www.mathworks.com/access/helpdesk/help/techdoc/matlab.html [retrieved 8 August 2007].

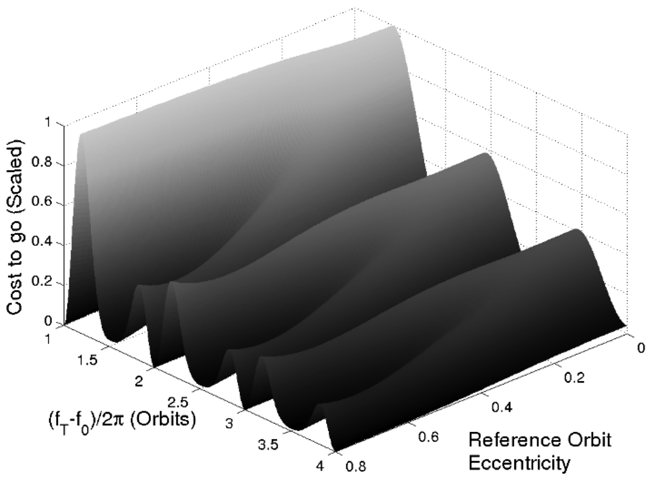

Fig. 1 The cost to go as a function of the eccentricity of the reference, and the final value of the true anomaly.

Hamiltonian provides a necessary condition for the maneuver with the smallest true anomaly change. In this case, solving this equation is particularly difficult due to its complicated nature, because the true anomaly expresses itself through various orders of harmonics. However, additional insight into the problem may be obtained by studying Fig. 1 which corresponds to a sample rendezvous from $\mathbf{x}_0 = [0; 1; 0; 0.5; 0; 1]^T$ to $\mathbf{x}_T = [0; 2; 0; 1; 0; 2]^T$ chosen with $f_0 = 35$ deg. The figure shows the value of the cost to go, scaled by the maximum cost, as a function of the eccentricity of the reference, and the final value of the true anomaly. The following features of the rendezvous cost are observed as follows:

1) The cost to go decreases as the final value of the true anomaly (and consequently, final time of the rendezvous) increases. This is a consequence of the fact that the matrix $\bar{N}$ comprises the growth term $K(f)$, and the cost is a function of $\bar{N}^{-1}$.

2) Depending on the final value of the true anomaly, the cost for rendezvous can be significantly high. For example, for a circular or near-circular orbit, it is best to wait for the completion of the target's orbit about the gravitational center, to obtain lower costs.

3) The behavior of the cost to go with changing eccentricity is an example of pitchfork bifurcation. For low eccentricities, it is observed that the rendezvous may be performed with minimal cost if the final time is such that the rendezvous is performed in intervals of complete orbits. However, for highly eccentric orbits, new minima appear approximately halfway between complete orbits. For this particular example, these correspond to regions near the apogee.

Two examples are presented to show the performance of the feedback scheme for near-optimal rendezvous.

Case A: Reference Orbit with Eccentricity $e = 0.5$

To ensure that arbitrary choices of the semimajor axis do not lead to physically meaningless orbits, the radius at perigee r_p is fixed and its value chosen for this case is 7100 km. The other parameters are chosen as $\rho_0 = 100$ Km, $f_0 = 0$, $f_T = 4\pi$, $\mathbf{x}_0 = [0; 1; 0; 0.5; 0; 1]^T$, and $\mathbf{x}_T = [0; 0; 0; 0; 0; 0]^T$. From the data given for this example, one can easily obtain the perturbation parameter to be $\varepsilon = 0.0094$. The gains are stored after performing the backward integration and the actual nonlinear system, Eq. (6), is propagated with the designed optimal feedback control law. Except for the cost to go, which is nondimensional, units for position and control acceleration used in this study are in kilometers and $\mathrm{m/s^2}$, respectively.

Figure 2a shows the relative trajectory in the rotating frame attached to the target; the circle marks the initial relative position of the satellite with respect to the target. The reference trajectory is shown using a dot-dashed line. The optimal trajectory accounting for nonlinear differential gravity is indicated by the dotted line and is obtained by applying the optimal feedback control law. The dashed line shows the result of applying the analytical solution for the control obtained from the linear problem with $\varepsilon = 0$. As an additional verification, the resulting trajectories obtained by using the feedback control law can be compared with the respective open-loop optimal trajectories, obtained by using a shooting method. In Fig. 2b, the solid line shows the scalar position error between the trajectory using the analytical control for the LQ problem and that obtained from the open-loop solution; the dashed line shows the same between the trajectories from the updated feedback law and the open-loop solution. The use of the updated feedback law results in a trajectory that is significantly different from that obtained using the analytical control law and it more closely matches the open-loop solution. The results shown were obtained by using 1000 data storage points. Furthermore, for the case with 100 gain data storage points and cubic spline interpolation, the terminal constraint is satisfied with an error of the order of 0.01 mm with the use of the updated feedback control law as compared to the error obtained from the analytical law, which is approximately 10 cm. The nonlinear effects can be discerned from the differences between the optimal trajectory and the reference solution. Optimal control histories for the open loop, feedback, and the reference solutions are shown in Fig. 2c. It is observed that the updated feedback control matches with the open-loop optimal solution quite well, whereas the reference control shows deviations, especially in the radial and along-track directions. It is also observed that most of the control is applied whenever the target crosses its apogee ($f = \pi$ and $f = 3\pi$). Figure 2d shows a comparison between the normalized cost to go for the feedback and open-loop solutions. It is clear from the scale of Fig. 2d that there is essentially no difference between the two results.

Case B: Reference Orbit with Eccentricity $e = 0.9$

As a second example, a reference orbit with $e = 0.9$ is chosen. The remaining parameters are $r_p = 7100$ km, $\rho_0 = 10$ km, $f_0 = 35\pi/180$, and $f_T = 2\pi$. The initial and final conditions for this example are $\mathbf{x}_0 = [0; 1; 0; 0.5; 0; 1]^T$ and $\mathbf{x}_T = [0; 2; 0; 1; 0; 2]^T$. Even though this example corresponds to $\varepsilon = 0.0016$, it should be noted that the term $(1 + e\cos f)$ in the denominator of Eqs. (10–12) will magnify the effects of nonlinearity for such a highly eccentric orbit. The relative trajectory using the control laws developed is shown in Fig. 3a. Figure 3b shows the trajectory distance errors between the analytical control law and the updated optimal feedback law, with respect to the open-loop solution, respectively. As indicated by the solid line, the deviation in the trajectory due to the use of the analytical law results in a terminal position error of more than 10 m. The resulting error using the optimal feedback law, as shown by the broken line, is approximately 50 mm, at the terminal point. This figure, therefore, clearly shows the advantage of using the near-optimal control law. In this case, as shown in Fig. 3c, the near-optimal feedback results are compared with those obtained from the open-loop solution. As is well known, convergence of the solution to the open-loop problem by using shooting methods is very sensitive to the initial guess. The feedback scheme not only helps in the correction of the reference solution, but it also provides the correct initial guess for intractable problems.

Conclusions

A near-optimal feedback control methodology is presented for minimum-fuel rendezvous near elliptic orbits accounting for nonlinear differential gravity. The rendezvous problem for the nonlinear differential gravity model is solved by the application of neighboring optimal feedback control methodology used in conjunction with a nominal trajectory, obtained by solving the related minimum-fuel feedback control problem for the linear Tschauner–Hempel equations, analytically. The analytical solution also provides remarkable insight into the cost of rendezvous and its dependence on true anomaly, especially for eccentric orbits. The process is facilitated through the use of a novel perturbation parameter, which simultaneously captures the effects of eccentricity and nonlinearity. Several numerical examples are analyzed to

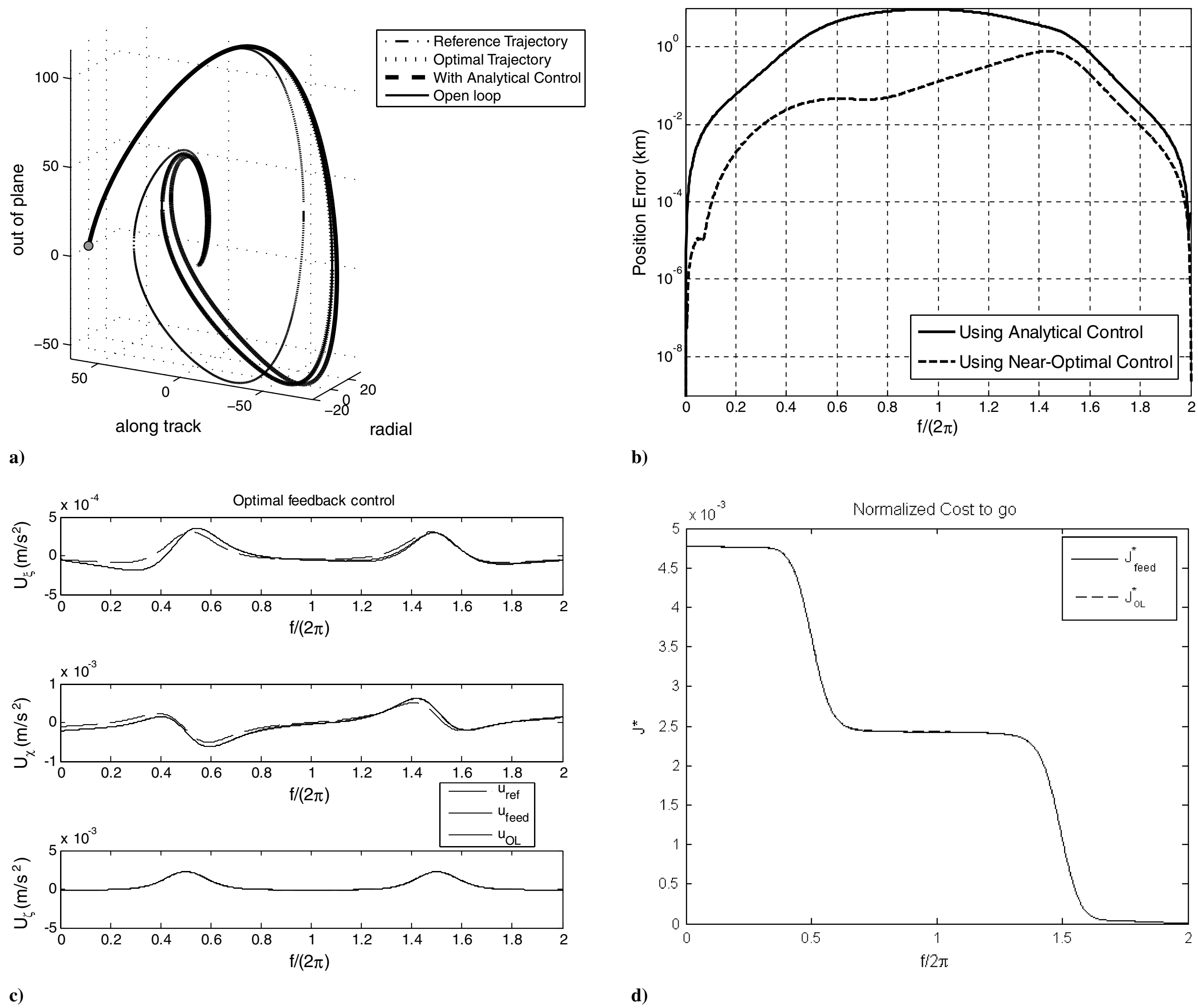

Fig. 2 a) Case A ($e = 0.5$) reference solution and the optimal relative trajectory (in km). b) Case A ($e = 0.5$) position errors obtained from analytical and the near-optimal feedback law. c) Case A ($e = 0.5$) reference, feedback, and open-loop optimal control histories. d) Case A ($e = 0.5$) comparison of nondimensional open loop and feedback cost to go.

demonstrate the efficacy of the method and the results are compared with those obtained from open-loop calculations.

Appendix A: Analytical Expressions of N's Elements

By denoting $r_i = 1/R_i$, the nonzero components of $N^{(0,\ldots,3)}(f)$ are

$$N_{44}^{(3)} = \frac{3}{2}\frac{e^2 r_1}{\eta^{15}} + 3\frac{r_2}{\eta^{13}}, \qquad N_{24}^{(3)} = eN_{44}^{(3)}, \qquad N_{22}^{(3)} = e^2 N_{44}^{(3)}$$

$$N_{44}^{(2)} = -\frac{3}{4}\frac{e^2 r_1}{\eta^{12}}\frac{\sin f(5e + 6\cos f + e\cos 2f)}{(1 + e\cos f)^3}$$

$$N_{34}^{(2)} = \frac{3}{2\eta^8}\left(\frac{e^2}{2\eta^2}r_1 + r_2\right)$$

$$N_{14}^{(2)} = -\frac{3e}{4\eta^{12}}(r_1 - 5r_2)$$

$$N_{24}^{(2)} = eN_{44}^{(2)}, \qquad N_{12}^{(2)} = eN_{14}^{(2)}, \qquad N_{22}^{(2)} = eN_{24}^{(2)}$$
$$N_{32}^{(2)} = eN_{34}^{(2)}$$

$$N_{66}^{(1)} = \frac{1}{8}\frac{(4 + 41e^2 + 18e^4)r_3}{\eta^{11}}, \qquad N_{55}^{(1)} = \frac{1}{8}\frac{(4 + 3e^2)r_3}{\eta^9}$$

$$\begin{aligned}
N_{44}^{(1)} = {} & \frac{r_1}{16\eta^{15}(1 + e\cos f)^4}\Bigg[e^3(32e^8 - 260e^6 + 273e^4 + 876e^2 \\
& + 80)\left(\frac{e}{8}\cos 4f + \cos 3f\right) + \frac{e^2}{2}(32e^{10} - 20e^8 - 1503e^6 \\
& + 2874e^4 + 5072e^2 + 552)\cos 2f + e(96e^{10} - 588e^8 \\
& - 541e^6 + 4296e^4 + 3296e^2 + 448)\cos f + \Bigg(12e^{12} \\
& + \frac{45}{2}e^{10} - \frac{6477}{8}e^8 + \frac{2503}{2}e^6 + 2583e^4 + 1256e^2 + 64\Bigg)\Bigg] \\
& + \frac{r_2}{8\eta^{13}(1 + e\cos f)^3}\Bigg[\frac{e^2}{2}(2e^6 - 41e^4 + 76e^2 \\
& + 40)\left(\frac{e}{2}\cos 3f + 3\cos 2f\right) + \frac{3e}{4}(2e^8 - e^6 - 184e^4 \\
& + 440e^2 + 128)\cos f + \frac{e^2}{2}(6e^6 - 39e^4 - 94e^2 + 512)\Bigg]
\end{aligned}$$

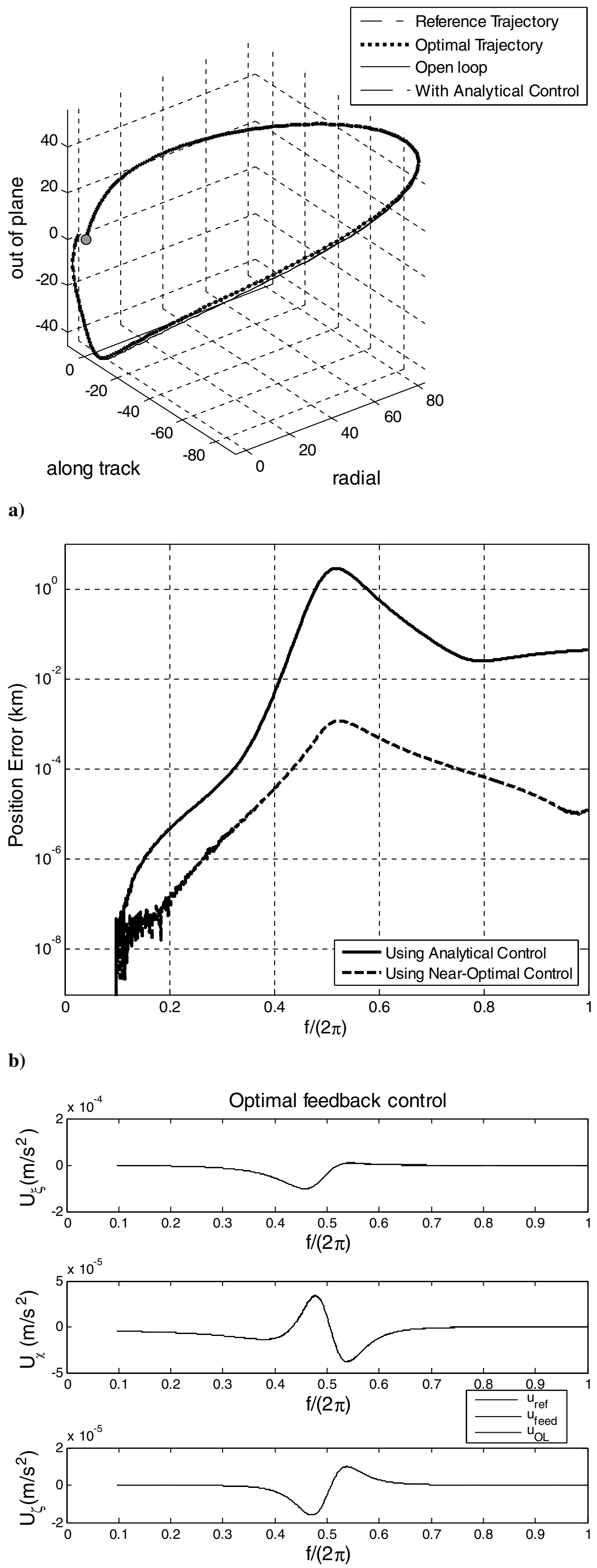

Fig. 3 a) Case B ($e = 0.9$) reference solution and the optimal relative trajectory (in km). b) Case B ($e = 0.9$) position errors obtained from analytical and the near-optimal feedback law. c) Case B ($e = 0.9$) reference, feedback, and open-loop optimal control histories.

$$N_{34}^{(1)} = \frac{\eta^5}{3} N_{44}^{(2)}, \qquad N_{33}^{(1)} = \frac{2\eta^5}{3} N_{34}^{(2)}$$

$$\begin{aligned}
N_{24}^{(1)} = &-\frac{e r_1}{16\eta^{15}(1+e\cos f)^4}\bigg[e(70e^8 - 113e^6 - 746e^4 \\
&- 220e^2 + 8)\left(\frac{e}{8}\cos 4f + \cos 3f\right) + \frac{1}{2}(46e^{10} + 403e^8 \\
&- 1544e^6 - 4672e^4 - 1264e^2 + 24)\cos 2f + e(178e^8 \\
&+ 101e^6 - 2946e^4 - 3516e^2 - 824)\cos f + \frac{1}{8}(114e^{10} \\
&+ 2237e^8 - 6662e^6 - 17{,}004e^4 - 12{,}504e^2 - 1216)\bigg] \\
&+ \frac{r_2}{8\eta^{13}(1+e\cos f)^3}\bigg[-\frac{e}{2}(13e^6 - 4e^4 - 66e^2 \\
&- 20)\left(\frac{e}{2}\cos 3f + 3\cos 2f\right) + \left(\frac{9}{4}e^8 - 60e^6 + \frac{123}{2}e^4 \right.\\
&\left.+ 237e^2 + 48\right)\cos f + \frac{e}{2}(e^6 - 94e^4 + 206e^2 + 272)\bigg]
\end{aligned}$$

$$N_{23}^{(1)} = \frac{e\eta^5}{3} N_{44}^{(2)}$$

$$\begin{aligned}
N_{22}^{(1)} = &-\frac{r_1}{16\eta^{15}(1+e\cos f)^4}\bigg[e^3(113e^6 + 536e^4 \\
&+ 360e^2 - 8)\left(\frac{e}{8}\cos 4f + \cos 3f\right) + \frac{e^4}{2}(137e^6 + 1094e^4 \\
&+ 3792e^2 + 1984)\cos 2f + e(339e^8 + 1996e^6 + 3416e^4 \\
&+ 1224e^2 + 32)\cos f + \left(-\frac{141}{8}e^{10} + 744e^8 + 1588e^6 \right.\\
&\left.+ 1793e^4 + 264e^2 + 8\right)\bigg] + \frac{r_2}{8\eta^{13}(1+e\cos f)^3}\bigg[-\frac{7e^2}{2}(4e^4 \\
&- 7e^2 - 8)\left(\frac{e}{2}\cos 3f + 3\cos 2f\right) + \frac{3e}{4}(4e^6 - 159e^4 \\
&+ 348e^2 + 192)\cos f + \left(-2e^6 - \frac{149}{2}e^4 + 253e^2 + 16\right)\bigg]
\end{aligned}$$

$$\begin{aligned}
N_{14}^{(1)} = &\frac{\sin f}{2\eta^9(1+e\cos f)^3}\bigg[3e(r_1 - 5r_2)\cos f \\
&+ \frac{e^2}{2}(r_1 - 5r_2)\cos 2f + \frac{e^2}{2}(5r_1 - r_2) - 12r_2\bigg]
\end{aligned}$$

$$N_{11}^{(1)} = \frac{1}{2}\frac{r_1}{\eta^9} + \frac{1}{8}\frac{(47e^2 + 16)r_2}{\eta^{11}}, \qquad N_{13}^{(1)} = \frac{2\eta^5}{3} N_{14}^{(2)}$$

$$N_{12}^{(1)} = e N_{14}^{(1)}$$

$$\begin{aligned}
N_{66}^{(0)} = &\frac{r_3 \sin f}{120\eta^8(1+e\cos f)^5}\bigg[\frac{e^2}{4}(190e^4 + 131e^2 - 6)\left(\frac{e}{2}\cos 4f \right.\\
&\left.+ 5\cos 3f\right) + \frac{e}{2}(150e^6 + 1789e^4 + 1271e^2 - 60)\cos 2f \\
&+ \left(\frac{1385}{2}e^6 + \frac{6365}{4}e^4 + \frac{2325}{2}e^2 - 60\right)\cos f + \frac{e}{8}(410e^6 \\
&+ 6993e^4 + 7282e^2 + 2640)\bigg]
\end{aligned}$$

$$N_{55}^{(0)} = \frac{r_3 \sin f}{120\eta^6(1+e\cos f)^5}\left[\frac{e^2}{4}(29e^2+6)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e}{2}(21e^4+269e^2+60)\cos 2f + \left(\frac{395}{4}e^4 + \frac{435}{2}e^2+60\right)\cos f + \frac{e}{8}(247e^4+478e^2+1200)\right]$$

$$N_{56}^{(0)} = -\frac{(5e^2-1)}{80\eta^{10}}\frac{(\cos f + e)r_3}{(1+e\cos f)^5}\left[\frac{e^3}{2}\cos 4f - e^2(e^2-5)\cos 3f + 2e(e^4-4e^2+10)\cos 2f + (-4e^6+17e^4-25e^2 + 40)\cos f + \frac{e(12e^6-49e^4+137e^2+40)}{2(5e^2-1)}\right.$$

$$N_{44}^{(0)} = -\frac{r_1 e \sin f}{240\eta^{12}(1+e\cos f)^5}\left[\frac{e^3}{4}(240e^6-2460e^4+5047e^2 + 2178)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e^2}{2}(21{,}300+46{,}687e^2 - 18{,}357e^4+180e^6+240e^8)\cos 2f + \frac{5e}{4}(-5556e^6 + 720e^8+31{,}098e^2+893e^4+15{,}888)\cos f + \left(7680 + \frac{80{,}197}{4}e^4 - \frac{76{,}939}{8}e^6 + \frac{795}{2}e^8 + 90e^{10} - 15{,}810e^2\right)\right] + \frac{r_2 e\sin f}{120\eta^{10}(1+e\cos f)^5}\left[\frac{e^3}{4}(10e^4+211e^2-606)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e^2}{2}(30e^6-289e^4-1591e^2+5700)\cos 2f + \frac{5e}{4}(82e^6-761e^4+198e^2+3792)\cos f - \left(\frac{65}{4}e^8 + \frac{47}{8}e^6 + 2250e^2 - \frac{4261}{4}e^4 + 1440\right)\right]$$

$$N_{33}^{(0)} = \frac{\eta^{10}}{9}N_{44}^{(2)}$$

$$N_{34}^{(0)} = -\frac{r_1 e}{12\eta^{10}(1+e\cos f)^4}\left[e^2(9e^4-20e^2-2)\left(\frac{e}{8}\cos 4f + \cos 3f\right) - \frac{e}{2}(6e^8-36e^6+11e^4+89e^2+21)\cos 2f + (-8e^8+67e^6-96e^4-30e^2-24)\cos f - \frac{e}{8}(24e^8 - 191e^6+208e^4+66e^2+348)\right] - \frac{r_2 e}{6\eta^8(1+e\cos f)^3}\left[\frac{e}{2}(12e^2-5)\left(\frac{e}{2}\cos 3f + 3\cos 2f\right) + \left(18e^4 + \frac{93}{4}e^2 - 3e^6 - 12\right)\cos f - \frac{e}{2}(10e^4-66e^2+21)\right]$$

$$N_{22}^{(0)} = -\frac{r_1 \sin f}{240\eta^{12}(1+e\cos f)^5}\left[\frac{e^2}{4}(1247e^6+3318e^4+480e^2 - 40)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e}{2}(1143e^8+12{,}947e^6 + 31{,}920e^4+4320e^2-280)\cos 2f + \left(\frac{19{,}865}{4}e^8 + \frac{34{,}695}{2}e^6 + 27{,}960e^4 + 3650e^2 - 120\right)\cos f + \frac{e}{8}(61e^8+61{,}414e^6+119{,}520e^4+84{,}520e^2+9760)\right] + \frac{r_2\sin f}{120\eta^{10}(1+e\cos f)^5}\left[\frac{e^2}{4}(220e^4-559e^2-46)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e}{2}(60e^6+1909e^4-5419e^2-400)\cos 2f + \left(385e^6 + \frac{2855}{4}e^4 - \frac{9995}{2}e^2 - 240\right)\cos f \frac{e}{8}(20e^6 + 5763e^4 - 11{,}278e^2 - 15{,}680)\right]$$

$$N_{23}^{(0)} = -\frac{r_1 e}{12\eta^{10}(1+e\cos f)^4}\left[e(3e^6-6e^4-12e^2+2) \times\left(\frac{e}{8}\cos 4f + \cos 3f\right) + \left(\frac{27}{2}e^6 - \frac{75}{2}e^4 - \frac{49}{2}e^2 + 3\right)\cos 2f + e(9e^6+2e^4-84e^2-18)\cos f\left(\frac{69}{8}e^8 - \frac{75}{4}e^6 + 14e^4 - \frac{255}{4}e^2 + 3\right)\right] - \frac{r_2}{6\eta^8(1+e\cos f)^3}\left[\frac{e}{2}(9e^4+3e^2-5) \times\left(\frac{e}{2}\cos 3f + 3\cos 2f\right) + \left(\frac{15}{4}e^6 + \frac{153}{4}e^4 - \frac{15}{4}e^2 - 12\right)\cos f + \frac{e}{2}(17e^4+57e^2-39)\right]$$

$$N_{24}^{(0)} = \frac{r_1 \sin f}{240\eta^{12}(1+e\cos f)^5}\left[\frac{e^3}{4}(730e^6-2347e^4 - 3348e^2-40)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e^2}{2}(450e^8+5107e^6 - 23{,}617e^4-31{,}410e^2-580)\cos 2f + \frac{5e}{4}(1886e^8 - 2513e^6-20{,}448e^4-21{,}248e^2-720)\cos f + \left(\frac{535}{4}e^{10} + \frac{25{,}239}{8}e^8 - 11{,}488e^6 - 16{,}600e^4 - 9130e^2 - 480\right)\right] + \frac{r_2\sin f}{120\eta^{10}(1+e\cos f)^5}\left[\frac{e^3}{4}(115e^4-174e^2-326)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e^2}{2}(15e^6+1099e^4-1914e^2-3050)\cos 2f + \frac{5e}{4}(113e^6+666e^4-2098e^2-1992)\cos f\left(-\frac{55}{8}e^8 + \frac{1429}{4}e^6 - \frac{689}{4}e^4 - 2105e^2 - 720\right)\right]$$

$$N_{11}^{(0)} = -\frac{r_1 \sin f}{12\eta^6(1+e\cos f)^3}(5e + 6\cos f + e\cos 2f) - \frac{r_2 \sin f}{120\eta^8(1+e\cos f)^5}\left[\frac{e^2}{4}(361e^2 - 46)\left(\frac{e}{2}\cos 4f + 5\cos 3f\right) + \frac{e}{2}(249e^4 + 3301e^2 - 400)\cos 2f + \left(\frac{4735}{4}e^4 + \frac{4885}{2}e^2 - 240\right)\cos f + \frac{e}{8}(443e^4 + 13,362e^2 + 3520)\right]$$

$$N_{12}^{(0)} = \frac{r_1}{12\eta^{12}(1+e\cos f)^4}\left[e(3e^6 - 6e^4 - 12e^2 + 2)\left(\frac{e}{8}\cos 4f + \cos 3f\right) + \frac{1}{2}(27e^6 - 75e^4 - 49e^2 + 6)\cos 2f + e(9e^6 + 2e^4 - 84e^2 - 18)\cos f + \left(\frac{69}{8}e^8 - \frac{75}{4}e^6 + 14e^4 - \frac{255}{4}e^2 + 3\right)\right] - \frac{r_2}{60\eta^{12}(1+e\cos f)^5}\left[\frac{e^2}{8}(75e^6 + 450e^4 + 75e^2 + 23)\left(\frac{e}{2}\cos 5f + 5\cos 4f\right) + \frac{5e}{16}(111e^8 + 966e^6 + 3711e^4 + 659e^2 + 160)\cos 3f + \left(270e^8 + \frac{2615}{2}e^6 + 2520e^4 + 515e^2 + 60\right)\cos 2f + \frac{5e}{8}(81e^8 + 1752e^6 + 5017e^4 + 5225e^2 + 1008)\cos f + \frac{1}{8}(1449e^8 + 10,370e^6 + 16,105e^4 + 10,845e^2 + 480)\right]$$

$$N_{13}^{(0)} = \frac{\eta^5}{3}N_{14}^{(1)}$$

$$N_{14}^{(0)} = \frac{r_1}{12\eta^{12}(1+e\cos f)^4}\left[e^2(9e^4 - 20e^2 - 2)\left(\frac{e}{8}\cos 4f + \cos 3f\right)\frac{e}{2}(6e^8 - 36e^6 + 11e^4 + 89e^2 + 21)\cos 2f + (-8e^8 + 67e^6 - 96e^4 - 30e^2 - 24)\cos f - \frac{e}{8}(24e^8 - 191e^6 + 208e^4 + 66e^2 + 348)\right] + \frac{r_2}{60\eta^{12}(1+e\cos f)^5}\left[-\frac{e^3}{8}(105e^4 + 355e^2 + 163)\left(\frac{e}{2}\cos 5f + 5\cos 4f\right) + \frac{5e^2}{16}(24e^8 - 261e^6 - 871e^4 - 3279e^2 - 1220)\cos 3f + \frac{5e}{2}(22e^8 - 248e^6 - 268e^4 - 1126e^2 - 249)\cos 2f + \frac{1}{8}(300e^{10} - 1135e^8 - 11,685e^6 - 14,825e^4 - 35,190e^2 - 2880)\cos f + \frac{e}{8}(656e^8 - 4995e^6 - 4505e^4 - 18,585e^2 - 11,820)\right]$$

Appendix B: Explicit Expressions of Hamiltonian's Derivatives with Respect to State and Costate

From the Hamiltonian defined by Eq. (50), it follows that

$$H_u = R\mathbf{u} + B^T\boldsymbol{\lambda} \Rightarrow H_{uu} = R, \qquad H_{u\lambda} = B^T$$

It can be shown that

$$H_\lambda = \mathbf{h}(\mathbf{x}, f) + B\mathbf{u}$$

Finally,

$$H_x = \left(\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right)^T\boldsymbol{\lambda} \Rightarrow H_{xu} = O, \qquad H_{x\lambda} = \left(\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right)^T$$

$$H_{xx} = \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right)^T\boldsymbol{\lambda}$$

Clearly, only $H_{x\lambda}$ and H_{xx} need to be provided explicitly. The nonzero entries of the partial derivative matrix of $\mathbf{h}(\mathbf{x}, f)$ along the reference trajectory are given below, with the notation, $d = [(1+\varepsilon x)^2 + \varepsilon^2 y^2 + \varepsilon^2 z^2]^{1/2}$:

$$\frac{\partial h_1}{\partial \nu_1} = 1$$

$$\frac{\partial h_2}{\partial \nu_2} = 1$$

$$\frac{\partial h_3}{\partial \nu_3} = 1$$

$$\frac{\partial h_4}{\partial x} = \frac{1}{(1+e\cos f)}\left(1 - \frac{1}{d^3} + 3\frac{(1+\varepsilon x)^2}{d^5}\right)$$

$$\frac{\partial h_4}{\partial y} = \frac{3}{(1+e\cos f)}\frac{(1+\varepsilon x)(\varepsilon y)}{d^5}$$

$$\frac{\partial h_4}{\partial z} = \frac{3}{(1+e\cos f)}\frac{(1+\varepsilon x)(\varepsilon z)}{d^5}$$

$$\frac{\partial h_4}{\partial \nu_2} = 2$$

$$\frac{\partial h_5}{\partial x} = \frac{3}{(1+e\cos f)}\frac{(1+\varepsilon x)(\varepsilon y)}{d^5}$$

$$\frac{\partial h_5}{\partial y} = \frac{1}{(1+e\cos f)}\left(1 - \frac{1}{d^3} + 3\frac{(1+\varepsilon x)^2}{d^5}\right)$$

$$\frac{\partial h_5}{\partial z} = \frac{3}{(1+e\cos f)}\frac{(\varepsilon x)(\varepsilon z)}{d^5}$$

$$\frac{\partial h_5}{\partial \nu_1} = -2$$

$$\frac{\partial h_6}{\partial x} = \frac{3}{(1+e\cos f)}\frac{(1+\varepsilon x)(\varepsilon z)}{d^5}$$

$$\frac{\partial h_6}{\partial y} = \frac{3}{(1+e\cos f)}\frac{\varepsilon^2 yz}{d^5}$$

$$\frac{\partial h_6}{\partial z} = \frac{1}{(1 + e\cos f)}\left(-e\cos f - \frac{1}{d^3} + 3\frac{\varepsilon^2 z^2}{d^5}\right)$$

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